EINSTEIN'S THEORY OF GRAVITATION: DETERMINATION OF THE FIELD BY LIGHT SIGNALS.

By Edward Kasner.

In this first paper we discuss the determination of a four-dimensional manifold

$$ds^2 = \Sigma g_{ik} dx_i dx_k \qquad (i, k = 1, 2, 3, 4) \tag{1}$$

obeying Einstein's equations of gravitation $G_{\mu\nu} = 0$, when we are given merely *the light equation*

$$\Sigma g_{ik} dx_i dx_k = 0.$$

This means that the ratios of the ten potentials g_{ik} are given and the problem is to determine the potentials themselves so that the gravitational equations are fulfilled.

The result shows that for example the solar gravitational field, or any field assumed to differ only slightly from the galilean field or flat space, can be completely explored by light signals alone. The corresponding determination by orbits alone will be given later. *The light rays determine the orbits, and vice versa.* A statement of results for both problems was given in *Science*, October 29, 1920 (vol. 52, pp. 413–14), in particular the connection between the planetary and light observations for the solar field.

We have also shown that the (exact) solar field can be regarded as immersed in a flat space of 6 dimensions; but that no solution of the Einstein equations can be obtained from flat space of 5 dimensions. (Cf. abstracts in *Bull. Amer. Math. Soc.*, vol. 27, pp. 62 and 102.)

§ 1. General Equations.

In order to write the general equations of gravitation we need the following symbolism.

Let g denote the determinant of fourth order formed from the coefficients $g_{\mu\nu}$; and let $g^{\mu\nu}$ denote the minor of the element $g_{\nu\mu}$ divided by g.

The Christoffel three-index symbol of the first kind is

$$[\alpha\beta, \gamma] = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x_\beta} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\gamma}\right); \tag{2}$$

and that of the second kind is

$$\{\alpha, \beta, \gamma\} = g^{\gamma\epsilon}[\alpha\beta, \epsilon], \tag{3}$$

where, by the regular convention of the Einstein literature, summation is to be understood, the repeated index ϵ stating the values 1, 2, 3, 4.

The Riemann-Christoffel curvature tensor is

$$(4)\quad B^{\rho}_{\mu\nu\sigma} = \{\mu\sigma, \epsilon\}\{\epsilon\nu, \rho\} - \{\mu\nu, \epsilon\}\{\epsilon\sigma, \rho\} + \frac{\partial}{\partial x_\nu}\{\mu, \sigma\rho\} - \frac{\partial}{\partial x_\sigma}\{\mu\nu, \rho\},$$

which is also denoted by the four-index symbol $\{\mu\rho, \sigma\nu\}$.

The "contracted" Riemann-Christoffel tensor (which might well be termed the Einstein tensor) is then

$$(5)\qquad G_{\mu\nu} = B^{\rho}_{\mu\nu\rho},$$

summation understood with respect to ρ.

This can be reduced to

$$(6)\qquad G_{\mu\nu} = \{\mu\alpha, \beta\}\{\nu\beta, \alpha\} - \frac{\partial}{\partial x_\alpha}\{\mu\nu, \alpha\} + \frac{\partial^2 L}{\partial x_\mu \partial x_\nu} - \{\mu\nu, \alpha\}\frac{\partial L}{\partial x_\alpha},$$

where summations are taken with respect to α and β, and where

$$L = \log \sqrt{-g}.$$

In the actual world with real coördinates, the determinant g is negative since the fundamental quadratic form has three negative dimensions and one positive dimension (as in the Lorentz-Minkowski world based on the affine-euclidean form

$$dt^2 - dx^2 - dy^2 - dz^2,$$

that is, the special relativity theory). The results which follow are not dependent on this reality assumption. We may write in general

$$L = \tfrac{1}{2}\log g,$$

since only the derivatives of L are involved.

Einstein's law of gravitation (in empty space, the only case we shall here consider) is expressed by the set of ten equations

$$G_{\mu\nu} = 0,$$

involving the second derivatives of the ten functions g_{ik}. [This set takes the place of the single equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0,$$

which is the basis of the Newtonian theory.]

Any manifold (1) obeying these equations will be said to be of the Einstein type. "Euclidean space" is a special case characterized by the

vanishing of the curvature tensor (4). More exactly we should here employ the term flat space since we must include the affine-euclidean as well as the ordinary euclidean species. Flat space thus means that the potentials g_{ik} in (1) can be reduced to constants, or that there is no permanent gravitation.

§ 2. EUCLIDEAN OR FLAT SPACE.

We now show that the light equation of an Einstein field cannot reduce to the simple form $dt^2 - dx^2 - dy^2 - dz^2 = 0$ *unless there is no permanent gravitation.*

For this purpose we use the more symmetric form

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = 0,$$

and prove that the curvature tensor necessarily vanishes.

We proceed to calculate the gravitational equations $G_{\mu\nu} = 0$ in the case where the manifold is of the form

$$ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2), \tag{7}$$

where λ is an unknown function of the four coördinates. The potentials are here

$$g_{11} = g_{22} = g_{33} = g_{44} = \lambda,$$

the remaining six, g_{12}, g_{13}, etc., all vanishing. The determinant g reduces to λ^4; therefore

$$g^{11} = g^{22} = g^{33} = g^{44} = \lambda^{-1},$$

the other six, g^{12}, etc., also vanishing.

We thus find immediately the first symbols

$$[11, 1] = \tfrac{1}{2}\lambda_1, \qquad [11, 2] = -\tfrac{1}{2}\lambda_2,$$

$$[12, 1] = \tfrac{1}{2}\lambda_2, \qquad [12, 3] = 0,$$

where subscripts denote partial derivatives; and the second symbols

$$\begin{aligned} \{11, 1\} &= N_1, \qquad \{11, 2\} = -N_2, \\ \{12, 1\} &= N_2, \qquad \{12, 3\} = 0 \end{aligned} \tag{8}$$

where, for convenience, we introduce

$$N = \tfrac{1}{2}\log\lambda. \tag{8'}$$

The function L occurring in the Einstein equations is then

$$L = \log\lambda^2 = 2\log\lambda = 4N.$$

Since our problem is symmetric in the four coördinates it will be sufficient to calculate G_{12} (case of unlike subscripts) and G_{11}, (case of like subscripts).

$$G_{12} = \{1\alpha, \beta\}\{2\beta, \alpha\} - \{12, \alpha\}_\alpha + L_{12} - \{12, \alpha\}L_\alpha.$$

The first term on the right is a summation of 16 terms, of which 10 vanish and 6 reduce each to N_1N_2; the second term is a sum of 4 terms contributing $-2N_{12}$; the third term is directly $4N_{12}$; the last term is a sum of 4, contributing $-8N_1N_2$. We have then

$$\begin{aligned} G_{12} &= 6N_1N_2 - 2N_{12} + 4N_{12} - 8N_1N_2 \\ &= 2N_{12} - 2N_1N_2. \end{aligned}$$

The like subscript case is a little more complicated:

$$\begin{aligned} G_{11} &= \{1\alpha, \beta\}\{1\beta, \alpha\} - \{11, \alpha\}_\alpha + L_{11} - \{11, \alpha\}L_\alpha \\ &= 4N_1^2 - N_2^2 - N_3^2 - N_4^2 - N_{11} + N_{22} + N_{33} + N_{44} \\ &\quad + 4N_{11} - 4N_1^2 + 4N_2^2 + 4N_3^2 + 4N_4^2 \\ &= 3N_{11} + N_{22} + N_{33} + N_{44} + 2(N_2^2 + N_3^2 + N_4^2). \end{aligned}$$

Hence the conditions (necessary and sufficient) that a manifold of the form

$$ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)$$

shall be an Einstein manifold (that is, $G_{\mu\nu} = 0$*) are*

$$\begin{aligned} &N_{12} - N_1N_2 = 0, \qquad \text{etc.,} \\ (9) \qquad &3N_{11} + N_{22} + N_{33} + N_{44} + 2(N_2^2 + N_3^2 + N_4^2) = 0, \qquad \text{etc.} \end{aligned}$$

This is a set of 10 partial differential equations of the second order in one unknown function $N = \frac{1}{2}\log\lambda$.

It remains to show that the manifolds thus obtained are euclidean. This can be done either by actually integrating the above set, or by proving that the curvature tensor vanishes.

The easiest way to integrate the set is to use the transformation

$$(10') \qquad N = -\log M.$$

This gives

$$N_1 = -\frac{M_1}{M}, \qquad N_{12} = \frac{M_1M_2 - MM_{12}}{M^2}, \qquad N_{11} = \frac{M_1^2 - MM_{11}}{M^2}.$$

Hence the transformed set is

$$\begin{aligned} &M_{12} = 0, \qquad M_{13} = 0, \qquad \text{etc.} \\ (10) \qquad &M_{11} = M_{22} = M_{33} = M_{44}. \end{aligned}$$

We easily find the general solution to be

$$(11) \quad M = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + a_1x_1 + a_1x_2 + a_3x_3 + a_4x_4 + a_5$$

with the condition

$$(11') \qquad a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4aa_5 = 0,$$

so that the result involves five arbitrary constants.

Since $N = \frac{1}{2}\log\lambda$, it follows that $\lambda = M^{-2}$; hence our manifold can be written

$$(12) \qquad ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2}{M^2}.$$

This can be reduced by linear transformation of the coördinates to one of the three forms

$$(13) \qquad \frac{\Sigma dx_i^2}{(\Sigma x_i^2)^2}, \qquad \frac{\Sigma dx_i^2}{(x_1 + ix_2)^2}, \qquad \Sigma dx_i^2,$$

which are found to be flat manifolds.

In second method we avoid the integration of set (9); and calculate instead the uncontracted curvature tensor $\{\alpha\beta, \gamma\delta\}$. Of these symbols it is sufficient to consider the types

$$(14) \qquad \begin{matrix} \{11, 12\}, & \{12, 21\}, & \{11, 23\}, \\ \{12, 31\}, & \{21, 31\}, & \{12, 34\}. \end{matrix}$$

The first, third, and sixth vanish identically. The others give conditions of the types

$$(15) \qquad \begin{matrix} N_{12} - N_1N_2 = 0, \qquad \text{etc.} \\ N_{11} + N_{22} + N_3^2 + N_4^2 = 0, \qquad \text{etc.} \end{matrix}$$

These are the conditions (necessary and sufficient) that

$$ds^2 = \lambda(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2), \qquad N = \tfrac{1}{2}\log\lambda,$$

be euclidean.

It is obvious a priori that when this set is fulfilled so is the former set (9), since of course euclidean space obeys Einstein's equations. It is not obvious that the converse also holds; but it is a fact since we can verify directly that each set of ten equations in N is linearly reducible to the other set.

Hence the only manifolds of the form (7) *which obey Einstein's gravitational equations are (affine) euclidean.*

We may state this in more geometric form by noting that the only four-dimensional spreads which can be conformally represented on four-dimensional flat space are precisely those of the above form.

Hence of all the four-dimensional spaces which are conformally representable on flat space, the only ones which are of the Einstein type are (affine) euclidean.

This means that when a conformal representation of an Einstein manifold on a flat space is possible, the manifold is isometric to flat space.

§ 3. NEARLY-EUCLIDEAN MANIFOLDS.

We proceed to generalize our result to curved Einstein spaces; namely, to show that the light equation determines the space, or that two conformally equivalent Einstein spaces are applicable. We shall here give the proof for the case where the Einstein spaces differ infinitesimally from flat space.

By a *nearly-euclidean* (or approximately flat) space we understand

$$ds^2 = \Sigma dx_i^2 + \Sigma h_{ik} dx_i dx_k, \tag{16}$$

where the h's denote 10 functions (of the coördinates x_1, x_2, x_3, x_4), which functions together with their derivatives are supposed to be small, so that their squares and products are neglected. We could write instead

$$\Sigma dx_i^2 + \epsilon \Sigma h_{ik} dx_i dx_k,$$

where ϵ is a small number and the h's are arbitrary finite functions; but the first notation is more convenient.

The curvature tensor and the Einstein tensor then reduce to *linear* differential expressions of the second order. Of the 10 Einstein equations it is sufficient to write those corresponding to G_{12} and G_{11}. These are (the last two subscripts denoting partial differentiation)

$$\begin{aligned} &h_{13,\,23} + h_{23,\,13} - h_{12,\,33} + h_{14,\,24} + h_{24,\,14} - h_{12,\,44} \\ &\qquad\qquad - h_{33,\,12} - h_{44,\,12} = 0, \\ &2(h_{12,\,12} + h_{13,\,13} + h_{14,\,14}) - h_{11,\,22} - h_{11,\,33} - h_{11,\,44} - h_{22,\,11} \\ &\qquad\qquad - h_{33,\,11} - h_{44,\,11} = 0. \end{aligned} \tag{17}$$

In deriving these it is sufficient to note that for the above ds^2 we have

$$\begin{aligned} g_{11} &= 1 + h_{11}, & g_{12} &= h_{12}, \\ g &= 1 + \Sigma h_{ii}, & L &= \tfrac{1}{2}\Sigma h_{ii}, \\ g^{11} &= 1 - h_{11}, & g^{12} &= -h_{12}, \end{aligned} \tag{17'}$$

with the three-index symbols (both kinds here have same values)

$$\begin{aligned} \{11, 1\} &= \tfrac{1}{2}h_{11,\,1}, & \{11, 2\} &= h_{12,\,1} - \tfrac{1}{2}h_{11,\,2}, \\ \{12, 1\} &= \tfrac{1}{2}h_{11,\,2}, & \{12, 3\} &= \tfrac{1}{2}(h_{13,\,2} + h_{23,\,1} - h_{12,\,3}). \end{aligned} \tag{17''}$$

§ 4. The Light Equation of a Nearly-Euclidean ds^2.

This is found by putting ds^2 equal to zero; that is

$$\Sigma dx_i^2 + \Sigma h_{ik} dx_i dx_k = 0. \tag{18}$$

If a second manifold determined by 10 new functions H_{ik} is to have the same light equation, that is, if

$$\Sigma dx_i^2 + \Sigma H_{ik} dx_i dx_k = 0 \tag{19}$$

is to differ from the former equation merely by a factor λ, this factor must obviously differ only slightly from unity, that is,

$$\lambda = 1 + \mu,$$

where μ is a small function (of the same order as the h's). This shows that

$$\begin{aligned} H_{ii} &= h_{ii} + \mu \qquad (i = 1, 2, 3, 4),\\ H_{ij} &= h_{ij} \qquad (i \neq j). \end{aligned} \tag{20}$$

Suppose now that both spaces are of the Einstein type. Our problem is to find the conditions on the function μ which follow from the fact that the functions h and also the functions H obey the linear equations of second order (17). Substituting and subtracting we find these 10 equations

$$\begin{gathered} \mu_{12} = 0, \qquad \text{etc.},\\ 3\mu_{11} + \mu_{22} + \mu_{33} + \mu_{44} = 0, \qquad \text{etc.} \end{gathered} \tag{21}$$

The last four equations show that

$$\mu_{11} = \mu_{22} = \mu_{33} = \mu_{44} = 0; \tag{21$'$}$$

hence all the second derivatives of μ vanish. It follows that the function is linear, say

$$\mu = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + b. \tag{22}$$

Our result is that if ds^2 defines any given nearly-euclidean Einstein space, then the only other possible nearly-euclidean Einstein spaces which have the same light equation are defined by

$$dS^2 = (1 + \mu) ds^2, \tag{23}$$

where μ is a linear function (22).

It remains to show that these spaces are equivalent (isometric) to the original (18). To do this, consider the general infinitesimal point transformation

$$X_i = x_i + \xi_i, \tag{24}$$

where the ξ's denote small functions of the four coördinates.

We have

$$dX_i = dx_i + \xi_{i,j}dx_j \tag{24'}$$

(summation respect to j understood); therefore

$$\begin{aligned} \Sigma dX_i^2 &= \Sigma dx_i^2 + 2\xi_{i,k}dx_i dx_k \\ &= \Sigma dx_i^2 + 2\xi_{1,1}dx_1^2 + \text{etc.} + 2(\xi_{1,2} + \xi_{2,1})dx_1 dx_2 + \text{etc.} \end{aligned} \tag{24''}$$

We first inquire when this reduces to $(1 + \mu)\Sigma dx_i^2$. The conditions are

$$\begin{gathered} \xi_{1,1} = \xi_{2,2} = \xi_{3,3} = \xi_{4,4} = \mu, \\ \xi_{1,2} + \xi_{2,1} = 0, \quad \text{etc.} \end{gathered} \tag{25}$$

We find that with the above value (22) of μ this set of 10 equations for the 4 unknown functions is consistent, the complete solution being

$$\begin{aligned} \xi_1 &= x_1\Sigma a_i x_i - \tfrac{1}{2}a_1\Sigma x_i^2 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 + bx_1 + B_1, \\ \xi_2 &= x_2\Sigma a_i x_i - \tfrac{1}{2}a_2\Sigma x_i^2 + A_{21}x_1 + A_{23}x_3 + A_{24}x_4 + bx_2 + B_2, \\ \xi_3 &= \\ \xi_4 &= \end{aligned} \tag{26}$$

where $A_{21} + A_{12} = 0$, etc., so that the result depends on 15 arbitrary constants. This is recognized as the general infinitesimal conformal transformations in flat space of four dimensions.

Apply this transformation to (16). The result, neglecting terms containing products of h's and ξ's, is found to be

$$(1 + \mu)\Sigma dx_i^2 + \Sigma h_{ik}dx_i dx_k;$$

which may be rewritten, in virtue of (20),

$$\Sigma dx_i^2 + \Sigma H_{ik}dx_i dx_k.$$

This shows that the two quadratic forms, one determined by the h's, the other by the H's of (20) are equivalent when μ has the linear form (22). We may therefore state the

THEOREM. *If two nearly-euclidean spaces both obey Einstein's equations and have the same light equation, they are necessarily equivalent* (*isometric*).

A geometric restatement would be:

If two nearly-euclidean spaces of the Einstein type are capable of conformal representation, they are necessarily isometric.

§ 5. DETERMINATION BY QUADRATURES.

Suppose we are given the light equation of some unknown Einstein space; how shall we find that space. *This can always be done by differentiations*

and quadratures. For suppose that the ten functions h_{ik} in the equation (18) do not satisfy, as they are given, the linear gravitational equations (17); the problem is then to find a function μ so that the ten functions H_{ik} found by (20) shall satisfy (17). The ten equations in μ thus obtained can be solved for the second derivatives μ_{ik}, which are thus expressed as known functions of the x's. The equations, by assumption consistent, can therefore be solved by quadratures. The result is determined up to an additive linear term which does not essentially affect the quadratic form ds^2 obtained.

§ 6. GENERAL EINSTEIN MANIFOLDS.

If we do not assume our manifolds to be nearly-euclidean, the discussion is of course more difficult since we have then to face the exact non-linear expressions (6). We shall confine ourselves here to the statement that we are lead to a set of ten non-linear equations of the second order for the unknown factor λ; these can be solved for the ten second derivatives in terms of the first derivatives; existence theorems then show that *the solution depends on not more than five arbitrary constants* (one of these is trivial, being merely a constant factor). In the special case of § 2 it was easy to show that the ∞^5 quadratic forms actually obtained were all equivalent. The general discussion will be left for a later paper.

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